Pressure spectrum and Multifractal Analysis.

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Now, we apply the thermodynamic formalizm to CER. First, we'll give a precise estimate of $|\pi(C(x_1,\dots,x_n))| =: R(x_1,\dots,x_n)$ in "pressure-friendly" terms.

Lemma (on radius). Let $\Phi := -\log|f'|$. $\exists\, C = C(J)$ such that
$$C^{-1} e^{S_n\Phi(x)} \le |R(x_1,\dots,x_n)| \le C e^{S_n\Phi(x)} \quad \forall x \in R(x_1,\dots,x_n).$$
$S_n\Phi := \sum_{k=0}^{n-1} \Phi \circ f^k$, assuming $\max|R_i|$ is small enough.

The proof uses Köbe distortion Thm:

Thm (weak form of Köbe, Egg-Yolk principle). Let $0<\lambda<1$. $\exists\, C(\lambda)>0$: for any conformal $\varphi: B(a,r)\to\mathbb{C}$
$$\forall x,y \in B(a,\lambda r): \; C(\lambda)^{-1}|\varphi'(a)| \le \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \le C(\lambda)|\varphi'(a)|.$$

Pf of Thm (idea). Normalize everything so that $a=0$, $\varphi(0)=0$, $\varphi'(0)=1$. The class of such functions is compact, so is $\{\varphi'\}$ for them. Thus on $\{|z|\le\lambda\}$ $|\varphi'| \le C_1$, for some C_1. This immediately imply the upper bound, by integration. For the lower bound, just use compactness again to find $x_n\to x$, $y_n\to y$, $\varphi_n\to\varphi$, $\frac{|\varphi_n(x_n)-\varphi_n(y_n)|}{|x_n-y_n|}\to 0$, which would imply $\varphi(x)=\varphi(y)$, or $\varphi'(x)=0$ if $x=y$. This contradicts compactness. ∎

In particular, $\dfrac{\max_{|x|=\lambda r}|\varphi(x)-\varphi(a)|}{\min_{|x|=\lambda r}|\varphi(x)-\varphi(a)|} \le C(\lambda)^2$.

Pf of Lemma.

For any $y\in J$, f is conformal in some $B(y,r(y))$. Thus, by compactness of J, $\exists\, r_0>0$: f is conformal in $B(y,r_0)$ $\forall y\in J$. Let now r is such that $B(f(y),r)\subset f(B(y,r_0))$ $\forall y\in J$. Such r exists because every non-constant analytic map is open, and J is compact. Assume $\max|R_i|<r/2$.

Observe that for any $x\in J$, $f(y)=x$, since f is conformal in $B(y,r_0)$, there is unique conformal branch of f^{-1} in $B(y,r)$ which maps x to y. Denote it f_y^{-1}.

Covering property. Observe that $f^{-1}(B(x,r)) = \bigcup_{f(y)=x} f_y^{-1}(B(x,r))$. Indeed, $\supset$ is obvious. If $x' = f(y')\in B(x,r)$, then $f_{y'}^{-1}$ is defined in $B(x',r)$, and maps x to some y. Then $f_{y'}^{-1}$ and f_y^{-1} agree in x, so, since f is injective in $B(y,r_0)$, and $|y-y'|<r_0$, so $y'\in B(y,r_0)$. Thus $f_y^{-1}(x')=y'\in f_y^{-1}(B(x,r))$. This shows that CER is a local cover.

Let us now consider $x\in R(x_1,\dots,x_n)$, $a := f^{n-1}(x)\in R_{x_n}$.
Let g be a branch of $(f^{n-1})^{-1}$ on $B(a,r)$, such that $g(a)=x$. Then $R(x_1,\dots,x_n) = g(R_{x_n})$. Thus $C(\tfrac{1}{2})^{-1}|g'(a)| \le \dfrac{|R(x_1,\dots,x_n)|}{|R(x_n)|} \le C(\tfrac12)|g'(a)|$.

Now $|g'(a)| = e^{S_n\Phi(x)}$, and we proved the lemma with $C = C(\tfrac12)\min_j|R(j)|$. ∎

Note that since $\Phi = -\log|f'|$ is a C^1 function and $|R(x_1,\dots,x_n)|$ decays exponentially, Φ is a C^δ function on J in the metric coming from X_A^δ.

We can now define pressure spectrum of (J,V,f) by
$P(t) = P(t\Phi)$. Observe that $P(0) = h_{top}(f) := h_{top}(T_L^A) > 0$. $P(t)$ is strictly decreasing because for large n, $S_n\Phi<0$ (by the first property). As $t\to\infty$, $P(t)\to-\infty$, as $t\to-\infty$, $P(t)\to\infty$. Thus $\exists!\, t_0>0$: $P(t_0)=0$.

Thm (Bowen) $\mathrm{Hdim}\, J = t_0$.

Pf Let first $t>t_0$. Then $\sum |R(x_1,\dots,x_n)|^t \le C \sum_{\substack{(x_1,\dots,x_n)\\ x\in R(x_1,\dots,x_n)}} e^{tS_n\Phi(x)}$.

Since $P(t\Phi)<\gamma<0$ for some γ, $\sum|R(x_1,\dots,x_n)|^t \le Ce^{n\gamma}$ for large n. Thus $H_t(J)=0$.

Let now $\mu=\mu_{t_0\Phi}$ be the equilibrium state. Then for some A,B,

$A \le \dfrac{\mu(R(x_1,\dots,x_n))}{e^{t_0 S_n\Phi}} \le B$, we have $\lim_{n\to\infty} \dfrac{\log\mu(|R(x_1,\dots,x_n)|)}{\log|R(x_1,\dots,x_n)|} = t_0$. $|t_0$

since $|R(x_1,\dots,x_n)| \asymp e^{S_n\Phi}$ (up to a constant).
Notice now that we can use a version of Billingsley's lemma for $R(x_1,\dots,x_n)$ to see that $\dim\mu = t_0$. Thus $\dim J = t_0$.

Let me give another explanation. Let us take $\delta < \dfrac{\mathrm{dist}(J,\partial V)}{2}$, and let n be the smallest number with
$$2C(\tfrac12)|(f')^{\circ n}(x)|\,\delta > r_0 \quad (f \text{ is conformal in } B(y,r_0) \text{ for } y\in J).$$
Then $f^{\circ n}$ is conformal (since, $|f^{\circ k}(B(y,\delta))| < r_0$, $k\le n-1$), and $B(f^{\circ n}(x), r_1) \subset f^{\circ n}(B(x,\delta)) \subset B(f^{\circ n}(x), r_2)$ for some r_1, r_2 which do not depend on δ or x. $m := \min_{y\in J}\mu(B(y,r_1))$, $M := \max_{y\in J}\mu(B(y,r_2))$. $m, M>0$, since μ is invariant and f is mixing. Then

$C^{-1} m\, e^{t_0 S_n\Phi} \le \mu(B(x,\delta)) \le CMe^{t_0 S_n\Phi}$, because μ is Gibbs, so it has Jacobian $e^{-t_0\Phi}$ and Φ is Hölder ... $S_n\Phi$ is also Hölder with same constant

δ or x. $m := \min_{y \in J} \mu(B(y, r_1))$, $M := \max_{y \in J} \mu(B(y, r_2))$. $m, M > 0$, since μ is invariant and f is mixing. Then
$C^{-1} m\, e^{t_0 S_n \varphi} \le \mu(B(x,\delta)) \le C M e^{t_0 S_n \Phi}$, because μ is Gibbs, so it has Jacobian $e^{-t_0 \Phi}$, and Φ is Hölder, so $S_n \Phi$ is also Hölder with some constant independent on n. Indeed, $|S_n \Phi(x) - S_n \Phi(y)| \le \sum |\Phi(f^{\circ k}(x)) - \Phi(f^{\circ k}(y))| \le C_{Höl}(|x-y|^\gamma + \frac{|x-y|^\gamma}{C^\gamma} + \dots) \le \frac{C_{Höl}}{C^\gamma}\left(\sum \frac{1}{2^\gamma}\right)|x-y|^\gamma$.
On the other hand, $\|f'\|_\infty r_0 > \delta e^{S_n \Phi} = |f^{\circ n}{}'(x)| \delta > r_0$, so $\delta \underset{\text{const}}{\asymp} e^{-S_n \Phi}$. Thus $\mu(B(x,\delta)) \asymp \delta^{t_0}$. ∎

Note that we proved a bit more. We constructed a measure μ such that $C^{-1} < \frac{\mu(B(x,\delta))}{\delta^{t_0}} < C$ for all $x \in J$ and all $\delta < 1$. Such a measure is called a *geometric measure.*

Lemma. If there is a geometric measure with exponent t_0 on a set J ($\mu(J) = 1 = \|\mu\|$), then $H\dim J = P\dim J = M\dim J = t_0$.

Pf. It is enough to prove that $\overline{M\dim} J \le t_0$, since by Billingsley, $H\dim J = t_0$. Let $(B(x_i, \delta))_{i=1}^{P(\delta, J)}$ be any packing. Then $\sum \delta^t \le C \sum \mu(B(x_i,\delta)) \le C$, i.e. $P(\delta, J) \le C \delta^{-t_0}$, thus $\overline{M\dim} J \le t_0$ ∎

This analysis can be taken further. What we just established is the following statement.

(on measures) **Lemma.** Let μ be a Gibbs measure for a Hölder function φ, $P(\varphi) = 0$. Then, for some $C > 0$, and for any $x \in J$,
$$C^{-1} \frac{\exp(-S_n \varphi(x))}{\exp(S_n \Phi(x))} \le \frac{\mu(B(x,\delta))}{\delta} \le C \frac{\exp(-S_n \varphi(x))}{\exp(S_n \Phi(x))}, \text{ where}$$

Def. Let ν be an ergodic f-invariant probability measure on J. $\lambda_\mu := \int \log|f'|\, d\mu$ is called *the Lyapunov exponent* of μ.

By ergodic Thm, μ-a.e., $\lambda_\mu := \lim_{n\to\infty} \frac{1}{n} \log |(f^{\circ n})'(x)|$. Thus $\lambda_\mu \ge \log \lambda > 0$. (in our case, since we consider ~~expanding~~ repellers).
By Lemma on radius, $\lim_{n \to \infty} \frac{1}{n} \log |R(x, \dots, x_n)| = -\lambda_\mu$ μ-a.e.

Thm. (Volume Lemma). Let m be a Gibbs state with Hölder continuous potential φ. Then m-a.e. $x \in J$, $\exists$
$$\lim_{\delta \to 0} \frac{m(B(x,\delta))}{\log \delta} = \frac{h_\mu}{\lambda_\mu},$$
where μ is the unique f-invariant measure strongly equivalent to m.

Note that in this case, $h_\mu = -\int \varphi\, d\mu + P(\varphi)$, $\lambda_\mu = \int \log|f'|\, d\mu$.
(since μ is the equilibrium measure).

Pf. By subtracting $P(\varphi)$, we can assume $P(\varphi) = 0$ - it does not change the class of Gibbs measures. Also, by strong equivalence, it is enough to prove everything for μ.
By the previous lemma, $\frac{\log \mu(B(x,r))}{\log r} \asymp \frac{-S_n \varphi(x)}{S_n \Phi(x)} = \frac{-\frac{1}{n} S_n \varphi(x)}{-\frac{1}{n} \log |f^{\circ n}{}'(x)|} \xrightarrow[\mu\text{-a.e.}]{n \to \infty} \frac{h_\mu}{\lambda_\mu}$ ∎

Remark The Theorem holds for any ergodic μ.

Proof. By Shannon-McMillan, μ-a.e., $\lim_{n \to \infty} \frac{1}{n} \log \mu(R(x, x_1, \dots x_n)) = h_\mu$.
Thus, μ-a.e., $\lim_{n \to \infty} \frac{\frac{1}{n} \log \mu(R(x_1, \dots, x_n))}{\frac{1}{n} \log |R(x_1, \dots, x_n)|} = \frac{h_\mu}{\lambda_\mu}$ ∎

Corollary. $\dim_\mu(x) = \overline{\dim}_\mu(x) = \frac{h_\mu}{\lambda_\mu}$ μ-a.e. Thus
$\dim \mu = \underline{\dim}\, \mu = P\dim \mu = \underline{P\dim}\, \mu = \frac{h_\mu}{\lambda_\mu}$.

Let us apply this to the measures μ_t, which are the equilibrium measures for $\Phi = -t \log f'$. Then
$$\dim \mu_t = \underline{\dim}\, \mu_t = P\dim \mu_t = \underline{P\dim}\, \mu_t = \frac{h_{\mu_t}}{\lambda_{\mu_t}} = \frac{t \int \log f'\, d\mu + P(t)}{\int \log f'\, d\mu_t} =$$
$$t + \frac{P(t)}{\lambda_{\mu_t}} = t + \frac{P(t)}{\lambda(t)}, \quad \lambda(t) := \lambda_{\mu_t}.$$
Let us also observe that $P(t)$ depends on t real-analytically.
Also $P(t+\varepsilon) \ge h_{\mu_t} + (t+\varepsilon)\int \Phi\, d\mu_t = P(t) + \varepsilon \int \Phi\, d\mu_t = P(t) - \varepsilon \int \log f'\, d\mu_t = P(t) - \varepsilon \lambda_{\mu_t}$. Thus $P'(t) \ge \lambda_{\mu_t}$. On the other hand,
$P(t) \ge h_{\mu_{t+\varepsilon}} + t \int \Phi\, d\mu_{t+\varepsilon} = P(t+\varepsilon) - \varepsilon \int \log f'\, d\mu_{t+\varepsilon}$, so $P'(t) \le \lim_{\varepsilon \to 0} \lambda_{\mu_{t+\varepsilon}} = \lambda_{\mu_t}$, since $-\log|f'|$ is continuous, and $\mu_{t+\varepsilon} \to \mu_t$ weak* as $\varepsilon \to 0$. Thus

$P(t) - \varepsilon\lambda_{\mu_t}$. Thus $P'(t) \geq \lambda_{\mu_t}$. On the other hand,
$P(t) \geq h_{\mu_{t+\varepsilon}} + t\int \Phi \, d\mu_{t+\varepsilon} = P(t+\varepsilon) - \varepsilon \int \log f' \, d\mu_{t+\varepsilon}$, so $P'(t) \leq \lim_{\varepsilon \to 0} \lambda_{\mu_{t+\varepsilon}} = \lambda_{\mu_t}$,
since $-\log|f'|$ is continuous, and $\mu_{t+\varepsilon} \to \mu_t$ weak* as $\varepsilon \to 0$. Thus

$$\boxed{\dim \mu_t = t - \frac{P(t)}{P'(t)}}.$$

Observe also that $P(t)$ is convex, as $\lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n(y)=x} e^{t S_n \varphi(y)}$ — convex functions of t.

Let us now perform the Multifractal Analysis.
Let ν be a measure on J.
Define $J_\alpha = \{x : \dim_\mu(x) = \alpha\}$. Cannot expect it to be non-empty, but for "good" measures, it will be.
Define the dimensional spectrum of ν by
$F_\nu(\alpha) := H\dim J_\alpha$.

By Billingsley's Lemma, $F_\nu(\alpha) \leq \alpha$.
Also, if $\mu(J_\alpha) > 0$, then $F_\nu(\alpha) = \alpha$. Thus, if $\underline{\dim}\,\mu \leq \alpha \leq \dim \mu$, $F(\alpha) = \alpha$.

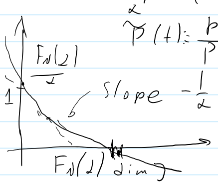
Let us now perform the Multifractal Analysis of $\nu := \mu_0$, the measure of maximal entropy.

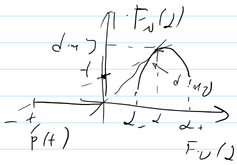
Thm. $F_\nu(\alpha)$ is a real-analytic function of α,

$$F_\nu(\alpha) = \inf_t \left(t + \alpha \tilde{P}(t) \right),$$
$$\tilde{P}(t) = \sup_\alpha \left(\frac{F_\nu(t) - t}{\alpha} \right).$$

— these two functions are Legendre-like transforms of each other.

$\tilde{P}(t) := \frac{P(t)}{P'(t)}$





To obtain $\tilde{P}(t)$, we draw the highest line through t intersecting the graph of $F_\nu(\alpha)$.

$F_\nu(\alpha) = -\infty \iff J_\nu(\alpha) = \emptyset$.

At t where inf reached, $1 + \alpha \tilde{P}'(t) = 0 \Rightarrow \tilde{P}'(t) = -\frac{1}{\alpha}$.

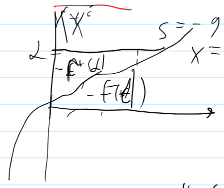
At α where sup is reached,
$$F_\nu'(\alpha) = -\frac{F_\nu(\alpha) - t}{t} = -\tilde{P}(t).$$

Let g be a decreasing function, defined on $\mathbb{R}$, g^{-1} be its inverse, defined on some $[\alpha_-, \alpha_+]$. Then let

$$f(t) := \int_0^t g(s)\,ds, \quad f^*(\alpha) := \int_0^\alpha g^{-1}(t)\,dt \text{ on } (\alpha_-, \alpha_+), \quad -\infty \text{ otherwise}$$

Then: Claim $\alpha t \geq f(t) + f^*(\alpha)$, with equality iff $\alpha = -f'(t)$.

Pf



The equality is reached iff $\alpha = -g(t) = -f'(t)$. ∎

Corollary. $f^*(\alpha) = \inf_t (\alpha s - f(t))$, $f(t) = \inf_\alpha (\alpha s - f^*(\alpha))$.
If f is real analytic, so is its Legendre transform.

If one consider $\tau := -S^{-1}$, i.e. $\tau(S(t)) = -t$. Then F and τ are related by

$$F(\alpha) = \inf_s (\alpha s - \tau(s))$$
$$\tau(s) = \inf_\alpha (\alpha s - F(\alpha))$$

Legendre transform.
Not quite symmetric, because $\tau(s)$ is defined for all s, $F(\alpha)$ only on $[\alpha_-, \alpha_+]$.

Pf. $J_\nu(\alpha) = \left\{x : \lim_{n\to\infty} \frac{\log \nu(R(x_1, \ldots, x_n))}{\log |R(x_1, \ldots, x_n)|} = \alpha \right\} = \left\{ x : \lim_{n\to\infty} \frac{P(0)}{\frac{1}{n} S_n \varphi(x)} = \alpha \right\} =$

$\left\{ x : \lim_{n\to\infty} \frac{1}{n} S_n \varphi(x) = \frac{P(0)}{\alpha} \right\}$.

Let us now take t_0 with $\tilde{P}'(t_0) = -\frac{1}{\alpha}$ $\left(P'(t) = -\frac{P(0)}{\alpha} \right)$, if such a t_0 exists.
Then μ_{t_0}-a.e., $\lim_{n\to\infty} \frac{1}{n} S_n \varphi(x) = -\lambda_{\mu_t} = -\frac{P(0)}{\alpha}$.
And $\dim \mu_{t_0} = \frac{h_\mu}{\lambda_\mu} = t_0 - \frac{P(t_0)}{P'(t_0)} = t_0 + \alpha \tilde{P}(t_0)$.
Thus $F_\nu(\alpha) \geq \dim \mu_t = t_0 + \alpha P(t_0)$.
Also, $\forall x \in J_\nu(\alpha)$, $\lim_{n\to\infty} \frac{\log \mu_{t_0}(R(x_1, \ldots, x_n))}{\log |R(x_1, \ldots, x_n)|} = \lim_{n\to\infty} \frac{t_0 S_n \varphi - P(t_0)}{S_n \varphi} = \alpha$,

Thus $F_\nu(\alpha) \geq \dim \mu_t = t_0 + \alpha P(t_0)$.

Also, $\forall x \in J_\nu(\alpha)$, $\lim\limits_{n\to\infty} \frac{\log \mu_{t_0}(R(x_1,\dots,x_n))}{\log |R(x_1,\dots,x_n)|} = \lim\limits_{n\to\infty} \frac{t_0 S_n\Phi - P(t_0)}{S_n\Phi} = \alpha$,

since on $J_\nu(\alpha)$, $\lim\limits_{n\to\infty} \frac{1}{n} S_n\Phi(x) = \frac{P(0)}{\alpha}$. Thus, by Billingsley's Lemma,

$H\dim J_\nu(\alpha) \leq F_\nu(\alpha)$.

If $\tilde{P}'(t) \neq -\frac{1}{\alpha}$ $\forall t$, then either $\tilde{P}'(t) > -\frac{1}{\alpha}$ $\forall t$ or $\tilde{P}'(t) < -\frac{1}{\alpha}$ $\forall t$.

Let $J_\nu(\alpha) \neq \emptyset$. Then $\forall x \in J_\nu$, $\lim\limits_{n\to\infty} \frac{1}{n} S_n\Phi(x) = \frac{P(0)}{\alpha}$.

Consider $\tilde{\mu}$ to be any weak-limit of the measures $\tilde{\mu}_k := \frac{1}{k}\sum_{j=0}^{k-1} \delta_{f^j(x)}$.

Then $\tilde{\mu}$ is f-invariant (because $\forall \varphi$ - cont, $\int \varphi \, d\tilde{\mu}_n - \int \varphi \circ f \, d\tilde{\mu}_n = \frac{1}{n}(\varphi(f^n(x)) - \varphi(x))$).

Thus $\tilde{\mu}$ can be decomposed into ergodic measures - this is called Choquet Thm. $\tilde{\mu} = \int \mu \, d\eta(\mu)$, μ - ergodic.

Thus $\tilde{\mu}$ a.e., $\lim\limits_{n\to\infty} \frac{1}{n} S_n\Phi(x) = -\frac{P(0)}{\alpha}$. Assume $\forall t$ $P'(t) > -\frac{1}{\alpha}$. Then

$\exists \mu$ - ergodic, μ-a.e. $\lim\limits_{n\to\infty} \frac{1}{n} S_n\Phi(x) > -\frac{P(0)}{\alpha}$ (for the case $< \frac{1}{\alpha}$, $K > \frac{P(0)}{\alpha}$)

But $\lim\limits_{n\to\infty} \frac{1}{n} S_n\Phi(x) = \int \Phi \, d\mu$. Observe now that $\forall t$: $\underbrace{t\int \Phi \, d\mu + h_\mu}_{h(t):=} \leq P(t)$.

Thus $-\frac{P(0)}{\alpha} > h'(t) = \int \Phi \, d\mu \geq \inf P'(t)$ - contradiction!

Remark. The same way, we can perform Multifractal Analysis of any Gibbs state m with Hölder potential φ.

Normalize φ so that $P(\varphi) = 0$ (i.e, in our previous situation, $\varphi = -P(0)$).

Then the role of $\tilde{P}(t)$ is played by the function $s(t)$, which is the unique solution of the equation

$P(s(t)\varphi - t\log|f'|) = 0 \iff P(s(t)\varphi + t\Phi) = 0$.

The Legendre-type transform is the same, with essentially the same proof. One just need to observe that $P(s,t) := P(s\varphi + t\Phi)$ is real-analytic, convex, and, for any fixed s, decreases from $+\infty$ to $-\infty$. Thus the inverse $t(s)$ is well-defined, convex, real-analytic. It can happen then that $s(t)$ is <u>not</u> defined for all t. So, sometimes people consider $\tau = -s^{-1}$, which was discussed above.